Lecture 6. Vector Spaces & Finite Dimensional Banach Spaces

<u>Plan</u> §1. Difinition of Vector Spaces and finite dim'l Vector Spaces

§2. The proof of "dimension is well defined".

part I. use linear algebras to reduce to a special case

part II use some tools from previous lectures.

§3. Some basic properties of dim.

§4. Finite dim'l Banach Spaces.

§1. Vector Spaces & finite dim'l Vector Spaces

<u>Notations</u> <u>Sym</u>. the category of sympathetic algebras

$U, V, \cdots$ vector spaces. $\mathbb{U}, \mathbb{V}, \cdots$ Vector Spaces.

Let $\mathfrak{S}$ be a sub-cat of <u>Ab</u> (e.g. $\mathbb{Q}_p$-vector sp. $\mathbb{Q}_p$-Banach sp)

Consider covariant functor $\mathbb{T}$: <u>Sym</u> $\longrightarrow \mathfrak{S}$. satisfying

(T1). $\mathrm{Spec}(\Lambda) \times \mathbb{T}(\Lambda) \longrightarrow \mathbb{T}(C)$

$(s, \lambda) \longmapsto \mathbb{T}(s)(\lambda)$ ($s(\lambda)$ is Colenz's language).

is continuous.

(T2) $\mathbb{T}(\Lambda) \longrightarrow \mathrm{Hom}_{cont}(\mathrm{Spec}(\Lambda), \mathbb{T}(C))$ is inj.

<u>Def</u>. take $\mathfrak{S}$ = <u>$\mathbb{Q}_p$-vector sp</u>, the cat of such functor $\mathbb{T}$ is called Vector Spaces.

...... <u>$\mathbb{Q}_p$-Banach sp.</u> .... Banach Spaces.

<u>Rmk</u>. without topological issue, Vector Spaces form an abelian cat.

so we can talk about ker, coker, im and exact sequences.

There are some basic and important examples

<u>Example</u> 1). $V \in \underline{\mathbb{Q}_p\text{-vector sp}}$ $\quad \underline{V} : \underline{\mathrm{Sym}} \longrightarrow \underline{\mathbb{Q}_p\text{-vector sp}}$

$$\Lambda \longmapsto V$$

$\underline{V}$ is a Vector Space (Banach Space if V admits a str of Banach alg)

2) $d \in \mathbb{N}$. $\quad \mathbb{V}^d : \underline{\mathrm{Sym}} \longrightarrow \underline{\mathbb{Q}_p\text{-Banach sp}}$

$$\Lambda \longmapsto \Lambda^d$$

$\mathbb{V}^d$ is a Banach Space / Vector Space.

3) $\mathbb{W}$. Vector Space. $\quad L \subset \mathbb{W}(C)$. sub $\mathbb{Q}_p$-vector sp.

we can define sub-space $\mathbb{L}$ of $\mathbb{W}$ by

$$\mathbb{L}(\Lambda) = \{ x \in \mathbb{W}(\Lambda) \mid \mathbb{W}(s)(x) \in L, \ \forall s \in \mathrm{Spec}(\Lambda) \}$$

<u>Finite dim'l Vector Spaces</u>

<u>Def</u>. Vector Space $\mathbb{W}$ is said to be of finite dim if $\exists$ exact sequences in <u>Vector Spaces</u>.

$$\begin{tikzcd} 0 \arrow[r] & \underline{V} \arrow[dr] & & & \\ 0 \arrow[r] & \underline{U} \arrow[r] & \mathbb{Y} \arrow[r] \arrow[dr] & \mathbb{V}^d \arrow[r] & 0 \\ & & & \mathbb{W} \arrow[r] & 0 \end{tikzcd}$$

where U & V are finite dim'l $\mathbb{Q}_p$-v.s.

we write $\underline{V} \to [\underline{U} \to \mathbb{Y} \to \mathbb{V}^d]$ for this presentation of $\mathbb{W}$

define the dim of $\mathbb{W}$ associate with this presentation to be

"$\dim \mathbb{W}$" $= (d, \dim_{\mathbb{Q}_p} U - \dim_{\mathbb{Q}_p} V)$

<u>Rmk</u>. for convenience, ~~in~~ in this talk, we say $\mathbb{W}$ has a pre of special type if $\exists$ a pre s.t $\mathbb{Y} = 0$. i.e. $[\underline{U} \to \mathbb{W} \to \mathbb{V}^d]$.

Main Thm dim is independent of presentation.

Prop* $\varphi: \mathbb{W} \to \mathbb{W}'$. morphism between finite dim'l Vector Spaces, then

1) $\ker\varphi$ is a f.d. Vector Space.

2) if φ is surj. $\mathbb{W}$ & $\mathbb{W}'$ admit pre of dim (d, a) & (d', a') resp.

then $\ker\varphi$ admits a pre of dim $(d-d', a-a')$

Proof of Prop* $\Rightarrow$ Thm take $\varphi = id_{\mathbb{W}}$. notice the dim of 0 must be $(0, 0)$

We have a special case of Prop*

Prop** $\mathbb{W}$: f.d. Vector Space admits a pre of special type. $[U \to \mathbb{W} \xrightarrow{\partial} \mathbb{V}^r]$

$\varphi: \mathbb{W} \to \mathbb{V}^1$. then

either φ is surj, $\ker\varphi$ is f.d. V.S. w/ pre of dim $(r-1, \dim U)$

or $\operatorname{Im}\varphi$ is f.d. $\mathbb{Q}_p$-v.s. $\ker\varphi$ is f.d.V.S. w/ pre of dim $(r, \dim U - \dim(\operatorname{Im}\varphi))$

Proof of Prop** $\Rightarrow$ Prop*.

Step 1 Prop* holds for $(\mathbb{W}, \underline{A})$ & $(\mathbb{W}, \mathbb{V}^1)$ (a little bit different statement) — $\underline{A}$: f.d. $\mathbb{Q}_p$.v.s

Step 2 Prop* holds for $(\mathbb{W}, \mathbb{V}^r)$

Step 3 Prop* holds for $(\mathbb{W}, \mathbb{W}')$ — special type.

Step 4 Prop* holds.

Step 1

Prop. $\mathbb{W}$. f.d.V.S. w/ a pre of dim (d, a).

1). A. f.d. v.s / $\mathbb{Q}_p$. $\varphi: \mathbb{W} \to \underline{A}$ surj. $\ker\varphi$. f.d.V.S. w/ a pre of dim $(d, a - \dim A)$.

2) $\varphi: \mathbb{W} \to \mathbb{V}^1$. $(\operatorname{Im}\varphi)(C)$ is not f.d / $\mathbb{Q}_p$.

the φ is surj, $\ker\varphi$ is f.d. V.S. w/ a pre of dim $(d-1, a)$.

Proof (1) compose with injection $A \hookrightarrow C$., we get mor

$\psi : \mathbb{W} \to \mathbb{V}^1$ w/ f.d image. so we only need to study (2)

(2). $\underline{V} \to [\underline{U} \to \mathbb{Y} \to \mathbb{V}^d]$. pre of $\mathbb{W}$., $\dim U - \dim V = a$.

$$\begin{tikzcd} \mathbb{Y} \arrow[r] \arrow[rr, bend right, "\tilde{\psi}"'] & \mathbb{W} \arrow[r, "\psi"] & \mathbb{V}^1 \end{tikzcd} \qquad \mathrm{Im}\,\tilde{\psi} = \mathrm{Im}\,\psi.$$

by Prop**. ① $\mathrm{im}\,\tilde{\psi}$ is f.d. $\ker\tilde{\psi}$ is f.d.V.S. w/ a pre of dim $(d-1, \dim U)$

② $\mathrm{im}\,\tilde{\psi}$ is not f.d. $\ker\tilde{\psi}$ is ------------- $(d, \dim U - \dim A)$

Consider exact seq $0 \to V \to \ker\tilde{\psi} \to \ker\psi \to 0$.

Lem $0 \to \underline{A} \to \mathbb{W}_1 \to \mathbb{W}_2 \to 0$ exact seq of V.S. A. f.d. v.s/ $\mathbb{Q}_p$.

for $\{i, j\} = \{1, 2\}$. $\mathbb{W}_i$ is f.d.V.S w/ a pre of dim (d_i, a_i)

$\Rightarrow \mathbb{W}_j$ is f.d.V.S. w/ a pre of dim (d_j, a_j)

where $d_1 = d_2$, $a_1 = a_2 + \dim A$.

(Lem ~~implies~~ completes the proof of Prop)

Proof of Lemma Exercise.

Step 2 Prop* holds for $\psi : \mathbb{W} \to \mathbb{V}^r$.

Proof induction on r. $r = 1$ ✓.

$\mathbb{V}^{r+1} \cong \mathbb{V}^r \oplus \mathbb{V}^1$. $\begin{tikzcd} \mathbb{W} \arrow[r, "\psi"] \arrow[rr, bend right, "\bar{\psi}"'] & \mathbb{V}^{r+1} \arrow[r] & \mathbb{V}^r \end{tikzcd}$ $\ker\bar{\psi}$ f.d.V.S. w/ a pre of dim $(d-r, a)$

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{W} \arrow[r, equal] \arrow[d, "\psi"] & \mathbb{W} \arrow[r] \arrow[d, "\bar{\psi}"] & 0 \\ 0 \arrow[r] & \mathbb{V}^1 \arrow[r] & \mathbb{V}^{r+1} \arrow[r] & \mathbb{V}^r \arrow[r] & 0 \end{tikzcd}$$

$\xRightarrow{\text{Snake lemma}}$ exact seq

$$0 \to \boxed{\ker\psi \to \ker\bar{\psi} \xrightarrow{f} \mathbb{V}^1} \to \mathrm{coker}\,\psi \to \mathrm{coker}\,\bar{\psi} \to 0$$

$\mathrm{coker}\,\psi = 0$ ψ surj

Step 1 $\Rightarrow \ker\psi = \ker\varphi$. f.d. ψ surj $\Rightarrow$ admit a pre of dim $(d-(r+1), a)$

Step 3 Prop** holds for $\psi: \mathbb{W} \to \mathbb{W}'$, $\mathbb{W}'$ admits a pre $0 \to \underline{U}' \to \mathbb{W} \to \mathbb{V}^{r} \to 0$.

Proof $\mathbb{W} \xrightarrow{\psi} \mathbb{W}' \to \mathbb{V}^{r}$, with the composite $\bar{\psi}$ — exact seq

$$0 \to \ker\psi \to \ker\bar{\psi} \to U' \to \operatorname{coker}\psi \to \operatorname{coker}\bar{\psi} \to 0$$

the same argument as step 2

Step 4 Prop* holds.

Proof $\underline{V}' \to [\underline{U}' \to \mathbb{Y} \to \mathbb{V}^{d'}]$ pre of $\mathbb{W}'$. $\dim U' - \dim V' = a'$

$\mathbb{Z} := \mathbb{Y}' \times_{\mathbb{W}'} \mathbb{W}$.

$$\begin{array}{ccccccccc} 0 & \to & V' & \to & \mathbb{Z} & \to & \mathbb{W} & \to & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \tilde{\psi}} & \ulcorner & \downarrow{\scriptstyle \psi} & & \\ 0 & \to & V' & \to & \mathbb{Y}' & \to & \mathbb{W}' & \to & 0 \end{array}$$

$\Rightarrow \ker\tilde{\psi} \cong \ker\psi$.

previous lemma $\left(0 \to \underline{A} \to \mathbb{W}_1 \to \mathbb{W}_2 \to 0 \Rightarrow d_1 = d_2,\ a_1 = a_2 - \dim A \right)$

$\Downarrow$

$\mathbb{Z}$ f.d.v.s w/ a pre of $\dim = (d + a + \dim V')$

apply step 3 to $\tilde{\psi}: \mathbb{Z} \to \mathbb{Y}'$, we are done.

Proof of Prop**

Idea Study elements in $\mathbb{W}(\widehat{C\{X\}})$, ~~reduced~~ reduce to $\mathbb{W} = \mathbb{L}_\ell$. — additive elements.

Recall $F := \operatorname{Frac}(C\{X\})$

(Shi Zhang's talk)

$$\begin{array}{ccc} \tilde{T}_C & \longrightarrow & \operatorname{Aut}(\bar{F}) \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{O}_C & \longrightarrow & \operatorname{Aut}(F) \\ a & \longmapsto & (X \mapsto X + a) \end{array}$$

$$1 \to \operatorname{Gal}(\bar{F}/F) \to \tilde{T}_C \to \mathcal{O}_C \to 1$$

$\operatorname{Gal}(\bar{F}/F) = \tilde{H}_{C\{X\}}$, $\quad \tau \mapsto x(\tau) := \tau(X) - X$.

$$\begin{array}{ccc} \tilde{T}_C \circlearrowright \widehat{B(0,1)} & & S_C \\ \downarrow & & \downarrow \\ \mathcal{O}_C \circlearrowright B(0,1) & & 0 \end{array}$$

S_C ← fixed element.

$\tau \in \tilde{T}_C$, $f \in \widehat{C\{X\}}$

define $f(\tau) := S_C(\tau(f)) \in C$.
$= \tau(S_C)(f)$

Recall (Yong Quan's talk)

Λ sympathetic alg. $T_\Lambda := \{\sigma \in \mathrm{Aut}_\Lambda(\widehat{\Lambda\{X\}}) \mid \chi(\sigma) : \sigma(X) - X \in \mathcal{O}_\Lambda\}$

$H_{\Lambda\{X\}} = \mathrm{Aut}(\Lambda\{X\}^{\infty}/\Lambda\{X\})$

exact seq. $1 \longrightarrow H_{\Lambda\{X\}} \longrightarrow T_\Lambda \longrightarrow \mathcal{O}_\Lambda \longrightarrow 1$

(Ji Quan's talk)

$\hat{\mathcal{E}} = \{f \in \widehat{C\{X\}} \mid \text{additive}\}$.

$\cup\!|$

$\mathcal{E} = \{f \in \hat{\mathcal{E}} \mid U_f^{\circ} := \{f_\infty(0)\} = f(\tilde{H}_{C\{X\}})$ is of finite rk$/\mathbb{Z}_p\}$

$S(\widetilde{\Lambda\{X\}}) = \{\psi \in \mathrm{Hom}(\widehat{\Lambda\{X\}}, \widehat{C\{X\}}) \mid \psi(X) = X,\ \psi(\Lambda) = C\}$

Some results on norms

Prop (CIC). Λ. spectral connected Banach C-alg, Λ'. normed C-alg.

$S \subset \mathrm{Hom}_{\mathrm{cont}}(\Lambda, \Lambda')$ satisfying

(*). $\sup_{\psi \in S} \|\psi(\lambda)\|_{\Lambda'} = \|\lambda\|_\Lambda, \quad \forall \lambda \in \Lambda.$

then for $\forall \lambda \in \Lambda$. TFAE

1) $\lambda \in C$

2) $\exists M \subset C$. compact. s.t. $\psi(\lambda) \in M$, $\forall \psi \in S$.

Prop. 1). $\forall f \in \widehat{\Lambda\{X\}}$. $\sup_{\tau \in T_\Lambda} \|f(\tau)\|_\Lambda = \|f\|_{\widehat{\Lambda\{X\}}}$

2) $\forall f \in \widetilde{\Lambda\{X\}}$. $\sup_{\psi \in S(\widetilde{\Lambda\{X\}})} \|\psi(f)\|_{\widehat{C\{X\}}} = \|f\|_{\widetilde{\Lambda\{X\}}}$.

Additive elements

Our goal is to define additive elements in $W(\widehat{C\{X\}})$ and study their properties.

Prop. $f \in \mathcal{T}$. Λ. sympathetic alg. then

1) $f(\tau\sigma) = f(\tau) + f(\sigma)$. $\forall \sigma, \tau \in T_\Lambda$

2) $\sigma(f) - f = f(\sigma) \in \Lambda$. $\forall \sigma \in T_\Lambda$

Proof. $\forall \tau \in T_\Lambda$, $\psi \in S(\widetilde{\Lambda\{x\}})$

$$\begin{tikzcd} C\{x\} \arrow[r, hook] \arrow[rrr, bend left, "\tau_\psi"] & \widetilde{\Lambda\{x\}} \arrow[r, "\tau"] & \widetilde{\Lambda\{x\}} \arrow[r, "\psi"] & \widetilde{C\{x\}} \end{tikzcd}$$

then $x(\tau_\psi) = \psi(x(\tau))$, $f(\tau_\psi) = \psi \circ \tau(f) - f$.

$\sigma, \tau \in T_\Lambda$. define $\beta := (\tau\sigma)_\psi\, \tau^{-1}_\psi, \sigma^{-1}_\psi, 0_\psi \in H_{C\{x\}}$

then $f(\beta) = f((\tau\sigma)_\psi) - f(\tau_\psi) - f(\sigma_\psi) + f(0_\psi)$

$= \psi(\tau\sigma(f) - \tau(f) - \sigma(f) + f) \in f(H_{C\{x\}})$, $\forall \psi$.

$$\xrightarrow[S = S(\widetilde{\Lambda\{x\}}),\ CIC]{\text{previous two props}} \tau\sigma(f) - \tau(f) - \sigma(f) + f \in f(H_{C\{x\}})$$

fix σ. $f_\sigma := \sigma(f) - f - f(\sigma)$. then $f_\sigma(\tau) = S_\Lambda(\tau\sigma(f) - \tau(f) - \sigma(f) + f)$

consider $\tau \mapsto f_\sigma(\tau)$. $\xrightarrow[CIC]{S = T_\Lambda}$ $f_\sigma \in f(H_{C\{x\}})$

$\Rightarrow f_\sigma(\tau) = f_\sigma(0) = 0$, $\forall \tau \in T_\Lambda$ $\xrightarrow{\cancel{S=T_\Lambda}}$ $f_\sigma = 0$.

$\|f\|_{\widetilde{\Lambda\{x\}}} = \sup_{\tau \in T_\Lambda} \|f(\tau)\|_\Lambda$.

i.e. $\sigma(f) - f = f(\sigma)$

for. (1). $f(\tau\sigma) - f(\tau) - f(\sigma) = S_\Lambda((\tau - 1)(f_\sigma)) = 0$.

apply to V.S. we have

Lem W. f.d. V.S. $l \in W(\widetilde{C\{x\}})$. TFAE.

1) $l(\tau\sigma) = l(\sigma) + l(\tau)$. $\forall \sigma, \tau \in \widetilde{T_C}$.

2) $l(0) = 0$. & $\sigma(l) - l \in W(C)$. $\forall \sigma \in \widetilde{T_C}$.

Proof 1) $\Rightarrow$ 2). $S_C(\tau(\sigma(l) - l)) = S_C(\tau\sigma(l) - \tau(l)) = l(\tau\sigma) - l(\tau) = l(\sigma)$

$\Rightarrow \sigma(l) - l = l(\sigma)$ by continuity.

2) $\Rightarrow$ 1). $\tau\sigma(l) - \tau(l) - \sigma(l) + l = (\tau - 1)(\sigma(l) - l) = 0$

$\xrightarrow{\text{apply } S_C}$ $l(\tau\sigma) = l(\sigma) + l(\tau)$.

Def. $l \in \mathbb{W}(\widetilde{\mathbb{C}\{x\}})$ is additive if it satisfies the conditions above.

l is additive of finite rk if $l(H_{\mathbb{C}\{x\}})$ is f.d $\mathbb{Q}_p$-v.s.

Lem $0 \to \underline{U} \to \mathbb{W}_1 \xrightarrow{\psi} \mathbb{W}_2 \to 0$ exact seq of f.d.v.s.

1). $l_1 \in \mathbb{W}_1(\widetilde{\mathbb{C}\{x\}})$ additive of f.rk $\Rightarrow$ so is $\psi(l_1) \in \mathbb{W}_2(\widetilde{\mathbb{C}\{x\}})$

2) $l_2 \in \mathbb{W}_2(\widetilde{\mathbb{C}\{x\}})$ additive of f.rk $\Rightarrow \exists !\ l_1$. add of f rk s.t $\psi(l_1) = l_2$.

Proof 1). l_1 add $\Rightarrow \psi(l_1)$ add.

2) uniqueness. $\psi(l_1) = \psi(l_1') = l_2$

$l_1 - l_1' \in U$. additive $\Rightarrow l_1 - l_1' = 0$

existence $\psi(l) = l_2$, $l_1 = l - l(0)$.

$\psi(\sigma(l_1) - l_1) = \sigma(l_2) - l_2 \in \mathbb{W}_2(\mathbb{C}) \Rightarrow \sigma(l_1) - l_1 \in \mathbb{W}_1(\mathbb{C})$.

U f.d $/\mathbb{Q}_p$ $\Rightarrow$ ~~f.rk~~ l_1 is f. rk

apply to general sympathetic alg. Λ

Prop $\mathbb{W}$. f.d. v.s. $l \in \mathbb{W}(\widetilde{\mathbb{C}\{x\}})$. add of f.rk.

$\forall\ \Lambda$ sympathetic alg, $\tau \in T_\Lambda$ we have $l(\tau\sigma) = l(\sigma) + l(\tau)$.

Proof. $\underline{V} \to [\underline{U} \to \mathbb{Y} \to \mathbb{V}^d]$. pre of $\mathbb{W}$.

$$\begin{tikzcd} \text{f.k add of } \mathbb{W} \arrow[r, leftrightarrow, "1:1"] \arrow[dr, bend right] & \text{f.k add of } \mathbb{Y} \arrow[r, leftrightarrow, "1:1"] & \text{f.k add of } \mathbb{V}^d \arrow[dl, bend left] \\ & \text{linear.} & \end{tikzcd}$$

$\rightsquigarrow$ assume $\mathbb{W} = \mathbb{V}^1$.

$f \in \mathbb{V}^1(\widetilde{\mathbb{C}\{x\}})$ add of f.rk $\Rightarrow f \in \mathfrak{C}$.

$\forall \tau \in T_\Lambda$. $l(\tau\sigma) - l(\sigma) - l(\tau) = \delta_\Lambda((\tau-1)(\sigma(l)-l)) = 0$.

Sub-Vector Spaces generated by additive elements

$\mathbb{W}$. f.d. V.S. $\ell \in \mathbb{W}(\widetilde{\mathbb{C}\{x\}})$. add of f.rk.

L_ℓ := sub-space of $\mathbb{W}(\mathbb{C})$ gen. by $\ell(T_{\mathbb{C}}) \rightsquigarrow \mathbb{L}_\ell$

(Recall. $\mathbb{L}_\ell(\Lambda) := \{\lambda \in \mathbb{W}(\Lambda) \mid s(\lambda) \in L_\ell,\ \forall s \in \mathrm{Spec}(\Lambda)\}$

Consider $\mathbb{W} = \mathbb{V}^2$. $(x, f) \in \mathbb{V}^2(\widetilde{\mathbb{C}\{x\}})$

$\rightsquigarrow L_{x,f}$, $\mathbb{L}_{x,f}$.

$L_{x,f} = \Gamma_f = \{(s(x), s(f)) \mid s \in \mathrm{Spec}(\widetilde{\mathbb{C}\{x\}})\}$

. ${}^t\mathbb{L}_{x,f^{-1}} = \mathbb{L}_{x,f}$.

Recall ~~([illegible])~~ $\overset{\circ}{U_f}$ $\qquad$ $\overset{\circ}{V_f}$

$$\overset{\shortparallel}{U_f} = \{f_{\omega,ap}(0)\}^\circ = f(\widetilde{H_{\mathbb{C}\{x\}}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_p. \qquad \overset{\shortparallel}{V_f} = \ker f_{\omega,ap} = \ker f_\omega \otimes_{\mathbb{Z}_p} \mathcal{O}_p$$

$\dim U_f = \dim V_f = \dim U_{tf} = \dim V_{tf}$.

exact seqs.

$$0 \to U_f \to L_{x,f} \xrightarrow{p_1} \mathbb{C} \to 0$$
$$0 \to V_f \to L_{x,f} \xrightarrow{p_2} \mathbb{C} \to 0$$

Prop. $f \in \mathcal{C}$, $p_1, p_2 : \mathbb{V}^2 \to \mathbb{V}^1$. projections.

then we have exact seqs

$$0 \to \underline{U_f} \to \mathbb{L}_{x,f} \xrightarrow{p_1} \mathbb{V}^1 \to 0$$
$$0 \to \underline{V_f} \to \mathbb{L}_{x,f} \xrightarrow{p_2} \mathbb{V}^1 \to 0.$$

Proof. ${}^t\mathbb{L}_{x,f^{-1}} = \mathbb{L}_{x,f}$. $\Rightarrow$ only need to prove the exactness of the first. one.

① Λ. sympathetic. $(\lambda, 0) \in \mathbb{L}_{x,f}(\Lambda)$ then $s(\lambda) \in U_f$, $\forall s \in \mathrm{Spec}(\Lambda)$

then $\lambda \in U_f$ by C.I.C. $\Rightarrow$ $\underline{U_f} = \ker p_1$

② $\lambda \in \mathcal{O}_\Lambda$. $\tau \in T_\Lambda$ s.t. $x(\tau) = \lambda$. $\underset{\mathbb{L}_{x,f}(\Lambda)}{\underset{\cap}{(x(\tau), f(\tau))}}$ preimage of λ.

Vector Spaces of dim = 1. (Proof of Prop **)

as add element in $\mathbb{V}^1(\widehat{C\{x\}})$

Lem $\mathbb{W}$. f.d. V.S. w/ pre $0 \to U \to \mathbb{W} \xrightarrow{\alpha} \mathbb{V}^1 \to 0$

$l \in \mathbb{W}(\widehat{C\{x\}})$ additive element. ~~corresponding~~ lifting to X, $U' = U \cap \mathbb{L}_l$. then

we have exact seq of V.S. $0 \to U' \to \mathbb{L}_l \xrightarrow{\alpha} \mathbb{V}^1 \to 0$.

Proof. only need to show $\mathbb{L}_l \xrightarrow{\alpha} \mathbb{V}^1$.

$\forall \lambda \in \mathcal{O}_\Lambda$. choose $\tau \in T_\Lambda$ s.t $x(\tau) = \lambda$. then $\alpha(l(\tau)) = \lambda$.

need to show $s(l(\tau)) \in L_l$. $\forall s \in \mathrm{Spec}(\Lambda)$

choose $\tilde{s} : \widehat{\Lambda\{x\}} \to \widehat{C\{x\}}$ s.t.

$$\begin{tikzcd} \widehat{\Lambda\{x\}} \arrow[r, "S_\Lambda"] \arrow[d, "\tilde{s}"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda \arrow[d, "s"] \\ \widehat{C\{x\}} \arrow[r, "S_C"] & C. \end{tikzcd}$$

commutes

$S_\lambda = S_C \circ \tilde{S} \circ \tau \big|_{\widehat{C\{x\}}} \in \mathrm{Spec}(\widehat{C\{x\}})$

$s\left(S_\Lambda(\tau(l))\right) = S_\lambda(l) \in L_l$.

Prop ** for $r = 1$

Proof $l \in \mathbb{W}(\widehat{C\{x\}})$ lifting X. $f = \psi(l) \in \mathcal{C}$.

case 1 $f = 0$ $\quad \mathbb{L}_l \subset \ker\psi$. $\Rightarrow$ $\ker\psi \xrightarrow{\alpha} \mathbb{V}^1$. by the previous lemma.

$(\ker\psi \cap U) \oplus U' \cong U \Rightarrow \mathbb{W} = \ker\psi \oplus U' \Rightarrow \mathrm{im}\,\psi \cong U'$.

hence we have exact seq. $0 \to U \cap \ker\psi \to \ker\psi \xrightarrow{\alpha} \mathbb{V}^1 \to 0$

$\dim_{\mathbb{Q}_p}(U \cap \ker\psi) = \dim_{\mathbb{Q}_p} U - \dim_{\mathbb{Q}_p}(\mathrm{Im}\,\psi)$

case 2 $f \neq 0$ $\quad (\alpha, \psi)(l) = (x, f)$

we have commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U' \arrow[r] \arrow[d] & \mathbb{L}_l \arrow[r, "\alpha"] \arrow[d, "(\alpha, \psi)"] & \mathbb{V}^1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U_f \arrow[r] & \mathbb{L}_{x,f} \arrow[r, "p_1"] & \mathbb{V}^1 \arrow[r] & 0 \end{tikzcd}$$

$\forall y \in U_f$. $\exists n \in \mathbb{N}$. $s \in \mathrm{Spec}(\widehat{C\{x\}})$

s.t $(0, y) = p^{-n} s(x, f) = (\alpha, \psi)(p^{-n} s(l))$

Hence $(\alpha,\psi) : \mathbb{L}_\ell \to \mathbb{L}_{x,f}$ is surj.

$\varphi \in \{\alpha, \psi\}$.
$$\begin{array}{ccccccccc} 0 & \to & \ker\varphi \cap \mathbb{L}_\ell & \to & \mathbb{L}_\ell & \xrightarrow{\varphi} & \mathbb{V}^1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \ker\varphi & \longrightarrow & \mathbb{W} & \to & \mathbb{V}^1 & \to & 0 \end{array}$$

Snake Lemma $\rightsquigarrow$ $0 \to (\ker\varphi)\cap\mathbb{L}_\ell \to \ker\varphi \to \mathbb{W}/\mathbb{L}_\ell \to 0$, $\mathbb{W}/\mathbb{L}_\ell \cong \mathbb{U}/\mathbb{U}'$

$\rightsquigarrow$ only need to study $(\ker\varphi)\cap\mathbb{L}_\ell$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{U}' & \to & \mathbb{L}_\ell & \xrightarrow{\alpha} & \mathbb{V}^1 & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle(\alpha,\psi)} & & \| & & \\ 0 & \to & \mathbb{U}_f & \to & \mathbb{L}_{x,f} & \xrightarrow{p_1} & \mathbb{V}^1 & \to & 0 \end{array}$$

$\rightsquigarrow$ $0 \to \ker(\alpha,\psi) \to \mathbb{U}' \to \mathbb{U}_f \to 0$

$\Rightarrow$ $\ker(\alpha,\psi)$. f.d.v.s. $/\mathbb{Q}_p$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{U}'' & \to & \mathbb{L}_\ell & \xrightarrow{\psi} & \mathbb{V}^1 & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle(\alpha,\psi)} & & \| & & \\ 0 & \to & \mathbb{V}_f & \to & \mathbb{L}_{x,f} & \xrightarrow{p_2} & \mathbb{V}^1 & \to & 0 \end{array}$$

$\rightsquigarrow$ $0 \to \ker(\alpha,\psi) \to \mathbb{U}'' \to \mathbb{V}_f \to 0$, $\mathbb{U}'' = (\ker\psi)\cap\mathbb{L}_\ell$

$\Rightarrow$ $(\ker\psi)\cap\mathbb{L}_\ell$. f.d.v.s.

$\dim(\ker\psi \cap \mathbb{L}_\ell) = \dim(\ker\alpha \cap \mathbb{L}_\ell)$

$\Rightarrow$ $\ker\psi$. f.d.v.s. $\dim(\ker\psi) = \dim\mathbb{U}' + \dim\mathbb{U}/\mathbb{U}' = \dim\mathbb{U}$.

<u>Proof of Prop*</u>.

$$0 \to \mathbb{U} \to \mathbb{W} \xrightarrow{\alpha} \mathbb{V}^r \to 0 \qquad \mathbb{D}_i \cong \mathbb{V}^1. \quad \mathbb{X}_i = \alpha^{-1}(\mathbb{D}_i)$$
$$\mathbb{W} = \bigoplus_{i=1}^r \mathbb{X}_i \qquad \mathbb{V}^r = \bigoplus_{i=1}^r \mathbb{D}_i.$$

$\rightsquigarrow$ reduce to the case $r = 1$ (exercise)

Additivity of dim

Prop. $\mathbb{W}' \hookrightarrow \mathbb{W}$. $\mathbb{W}'' = \mathbb{W}/\mathbb{W}'$ — f.d.V.S, then

$\mathbb{W}''$ is f.d.V.S. $\dim \mathbb{W}'' = \dim \mathbb{W} - \dim \mathbb{W}'$.

Proof step 1. $\mathbb{W}' \hookrightarrow \mathbb{V}^n$.

step 2 $\mathbb{W}' \hookrightarrow \mathbb{W}$ — special type.

step 3. general case (exercise)

We'll use the following lemma

Lem $\mathbb{W}$. f.d.V.S. $\dim = (d, a)$. $\mathbb{W} \hookrightarrow \mathbb{V}^n$. $\mathbb{V}^1 \cong \mathbb{D}_j$

$\exists\, J \subset \{1, 2, \dots, n\}$ $|J| = d$ $p: \mathbb{V}^n \to \bigoplus_{j \in J} \mathbb{D}_j$. proj

s.t. $p|_{\mathbb{W}}$ is surj, $\mathbb{W} \cap \bigoplus_{j \notin J} \mathbb{D}_j$ is f.d.V.S of dim $(0, a)$.

Proof use induction on d and Prop*.

Finite dimensional Banach Spaces

Prop. $\varphi: \mathbb{W}_1 \to \mathbb{W}_2$. f.d.B.S. then $\ker\varphi$ & $\operatorname{im}\varphi$ are f.d.B.S.

Proof. Let $\mathbb{X} \subset \mathbb{W}$ be a f.d. sub-V.S of a f.d.B.S.

we need to show $\mathbb{X}$ is sub-B.S. of $\mathbb{W}$.

Similarly, we can reduce to $\mathbb{W}$ admitting a pre $0 \to \mathbb{U} \to \mathbb{W} \xrightarrow{\varphi} \mathbb{V}^d \to 0$.

$\cancel{0 \to \mathbb{W}' \to \mathbb{W} \to \mathbb{V}^d}$

$\cancel{\mathbb{U}' \to \mathbb{X} \to \mathbb{V}^r}$

$\exists\, \mathbb{V}^r \subset \mathbb{V}^d$. $p: \mathbb{V}^d \to \mathbb{V}^r$ projection.

$p|_{\mathbb{X}}$ is surj ($\dim \mathbb{X} = (r, \ )$)

$0 \to \mathbb{U}' \to \mathbb{X} \xrightarrow{\bar\varphi} \mathbb{V}^r \to 0$.

$(h_1, \dots, h_r) \mapsto (X_1, \dots, X_r)$

odd elements

$(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{X}(\Lambda)$. limit point $\lambda \in \mathbb{W}(\Lambda)$.

$\exists k$. s.t. $\bar{\psi}(\lambda_n) \in (p^{-k}\mathcal{O}_\Lambda)^r$. $\forall n$.

then $\exists (\tau_{i,n})_{1\le i\le r} \in T_\Lambda^r$. s.t $\bar{\psi}(\lambda_n) = (p^{-k}\chi(\tau_{i,n}))_{1\le i\le r}$

$\tau_{i,n} \longrightarrow \tau_i \in T_\Lambda$ (choose a sub-sequence)

Let $\lambda' = \sum_i l_i(\tau_i)$, $\lambda'_n = \sum_i l_i(\tau_{i,n})$. $\mu = \lambda' - \lambda$.

since $\lambda' \in \sum_i \mathbb{L}_{l_i}(\Lambda) \subset \mathbb{X}(\Lambda)$.

only need to show $\mu \in U' \subset \mathbb{X}(\Lambda)$

$\mu_n = \lambda'_n - \lambda_n$. $s(\mu_n) \longrightarrow s(\mu)$ $\forall s \in \mathrm{Spec}(\Lambda)$

$\Rightarrow s(\mu) \in U'$, $\forall s \in \mathrm{Spec}(\Lambda)$ by continuity

the following lemma complètes the proof

<u>Lem</u> $\mathbb{W}$. f.d. V.S. $U' \subset \mathbb{W}(\mathbb{C})$. f.d. v.s / $\mathbb{Q}_p$. Λ sympathetic alg. $\lambda \in \mathbb{W}(\Lambda)$. TFAE

1) $s(\lambda) \in U'$. $\forall s \in \mathrm{Spec}(\Lambda)$

2) $\lambda \in U'$.

<u>Proof</u>.

$$\begin{tikzcd}
U'' \arrow[rr] \arrow[dr] \arrow[dd] & & U''' \arrow[d] \\
& \Gamma\mathbb{Y} \arrow[r] \arrow[d] & \mathbb{V}^d \\
U' \arrow[r] & \mathbb{W} &
\end{tikzcd}$$

$\Gamma\mathbb{Y} \ni \tilde{\lambda} \longmapsto (f_i)_{1\le i\le d} \in \mathbb{V}^d$, $\tilde{\lambda} \longleftrightarrow \lambda \in \mathbb{W}$

$\left[\begin{array}{l} \text{consider C.I.C. with} \\ S = \mathrm{Spec}(\Lambda). \quad M = U''' \cap \prod_{i=1}^{d} B(0, \|f_i\|_{sp}) \end{array} \right]$.